Interpolation with Unequal Intervals:

Lagrange's Interpolation Formula:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\dots(x_0-x_n)}(y_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}(y_1) + \dots\dots$$

$$\frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}(y_n).$$

Problems:

① Using Lagrange's interpolation formula, find the value of y corresponding to $x=10$ from the following data:

x :	5	6	9	11
y :	12	13	14	16

Sol

Given $x_0 = 5$ $x_1 = 6$ $x_2 = 9$ $x_3 = 11$,

$y_0 = 12$ $y_1 = 13$ $y_2 = 14$ $y_3 = 16$,

By Lagrange's interpolation Formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

To find y:

using $x=1$ and the given data

$$y(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13)$$

$$+ \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$$y(10) = 2 - 4.3334 + 11.6667 + 5.3334$$

$$y(10) = 14.6667$$

$$\boxed{y(10) \simeq 14.67}$$

(2) Apply Lagrange's formula, to find $f(5)$, given that $f(1)=2$, $f(2)=4$, $f(3)=8$ and $f(7)=128$

So

Given data

x :	1	2	3	7
$y=f(x)$:	2	4	8	128

$x_0 = 1$, $x_1 = 2$, $x_2 = 3$ $x_3 = 7$

$y_0 = 2$, $y_1 = 4$ $y_2 = 8$ $y_3 = 128$

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

Substituting $x=5$ and the given data.

$$f(5) = \frac{(3)(2)(-2)}{(-1)(-2)(-6)}(2) + \frac{(4)(2)(-2)}{(1)(-1)(-5)}(4) + \frac{(4)(3)(-2)}{(2)(1)(-4)}(8) + \frac{(4)(3)(2)}{(6)(5)(4)}(128)$$

$$\boxed{f(5) = 32.93}$$

(3) Using Lagrange's interpolation formula, Fit a polynomial of degree 3 for the following data and also find $f(2.5)$.

x :	-1	0	2	3
y :	-2	-1	1	4

Sol Given data

$x_0 = -1, \quad x_1 = 0, \quad x_2 = 2, \quad x_3 = 3$

$y_0 = -2, \quad y_1 = -1, \quad y_2 = 1, \quad y_3 = 4$

By Lagrange's Interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

Substituting the given data, we have

$$y = f(x) = \frac{x(x-2)(x-3)}{(-1)(-3)(-12)}(-2) + \frac{(x+1)(x-2)(x-3)}{(1)(-2)(-3)}(-1)$$

$$+ \frac{(x+1)(x)(x-3)}{(3)(2)(-1)}(1) + \frac{(x+1)x(x-2)}{(4)(3)(1)}(4)$$

$$y = f(x) = \frac{1}{6}[x^3 - 5x^2 + 6x] - \frac{1}{6}[x^3 - 4x^2 + x + 6]$$

$$-\frac{1}{6}[x^3 - 2x^2 - 3x] + \frac{1}{3}[x^3 - x^2 - 2x]$$

$$\boxed{f(x) = \frac{1}{6}[x^3 - x^2 + 4x - 6]}$$

This is required polynomial.

To find $f(2.5)$:

$$f(2.5) = \frac{1}{6}[(2.5)^3 - (2.5)^2 + 4(2.5) - 6]$$

$$= 2.23$$

(4) Using Lagrange's formula, Given $u_0 = 6$, $u_1 = 9$, $u_3 = 33$ and $u_7 = -15$. Find u_2.

(or)

Given

x :	0	1	2	3	7
y :	6	9	-	33	-15

Find the Missing term from the Table

Sol Given

$x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 7$.

$y_0 = 6$, $y_1 = 9$, $y_2 = 33$, $y_3 = -15$

where $y = u_x$.

To find y, when $x = 2$

By Lagrange's formula,

$$y = u(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}(y_0) + \ldots\ldots$$

$$y = u(x) = \frac{(x-1)(x-3)(x-7)}{(-1)(-3)(-7)}(6) + \frac{x(x-3)(x-7)}{1(-2)(-6)}(9)$$

$$+ \frac{x(x-1)(x-7)}{(3)(2)(-4)}(33) + \frac{x(x-1)(x-3)}{(7)(6)(4)}(-55)$$

putting $x = 2$, we have

$$y = u(2) = u_2 = -\frac{10}{7} + \frac{15}{2} + \frac{55}{4} + \frac{5}{28}$$

$$\therefore \boxed{u_2 = 20}$$

The missing value of y at $x = 2$ is 20.

HW

(1) Use Lagrange's formula, to find the value of y at $x = 6$, given the data.

x :	3	7	9	10
y :	168	120	72	63

Ans: $y(6) = 147$.

(2) Fit a polynomial to the following table

x :	0	1	3	4
y :	-12	0	6	12

Ans: $f(x) = x^3 - 7x^2 + 18x - 12$

(3) Find $y(9.5)$ given the data

x :	7	8	9	10
y :	3	1	1	9

Ans: $y(9.5) = 3.625$

Inverse Lagrange's Interpolation Formula:

$$x = f(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}(x_0) +$$

$$\frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}(x_1) +$$

$$\cdots\cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}(x_n)$$

—— x ——

Problems

(1) Apply Lagrange's formula Inversely to obtain the root of the equation $f(x)=0$ given that $f(0)=-4$, $f(1)=1$, $f(3)=29$ and $f(4)=52$.

Sol

Given that.

x:	0	1	3	4
y:	-4	1	29	52

$x_0 = 0, \quad x_1 = 1, \quad x_2 = 3, \quad x_3 = 4$

$y_0 = -4, \quad y_1 = 1, \quad y_2 = 29, \quad y_3 = 52$

To find 'x' such that $f(x)=0$.

(ie) $y = 0$.

Apply inverse Lagrange's formula

$$x = f(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}(x_0) + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}(x_1)$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}(x_2) + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}(x_3)$$

Using the given data and $y = 0$, we have

$$x = \frac{(-1)(-29)(-52)}{(-5)(-23)(-56)}(0) + \frac{(4)(-29)(-52)}{(5)(-28)(-51)}(1)$$

$$+ \frac{(4)(-1)(-52)}{(33)(28)(-23)}(3) + \frac{(4)(-1)(-29)}{(56)(51)(23)}(4)$$

$$x = 0.8448 - 0.0294 + 0.0071$$

$$\boxed{x = 0.8225}$$

HW

① Given the data

x :	3	5	7	9	11
y :	6	24	58	108	174

Find the value of x corresponding to $y = 100$

Ans: $x = 8.656$.